

Global boundedness of solutions in a parabolic-parabolic chemotaxis system with singular sensitivity

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Abstract

We consider a parabolic-parabolic Keller-Segel system of chemotaxis model with singular sensitivity $u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v)$, $v_t = k \Delta v - v + u$ under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), with $\chi, k > 0$. It is proved that for any $k > 0$, the problem admits global classical solutions, whenever $\chi \in (0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}})$. The global solutions are moreover globally bounded if $n \leq 8$. This shows an exact way the size of the diffusion constant k of the chemicals v effects the behavior of solutions.

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1 Introduction

In this paper, we consider the parabolic-parabolic chemotaxis system with singular sensitivity

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega, \quad t \in (0, T), \\ v_t = k \Delta v - v + u, & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), & x \in \Omega, \end{cases} \quad (1.1)$$

where $\chi, k > 0$, Ω is a smooth bounded domain in \mathbb{R}^n ($n \geq 2$), $\frac{\partial}{\partial \nu}$ denotes the derivation with respect to the outer normal of $\partial\Omega$, and the initial datum $u_0 \in C^0(\overline{\Omega})$, $u_0(x) \geq 0$ on $\overline{\Omega}$, $v_0 \in W^{1,q}(\Omega)$ ($q > n$), $v_0(x) > 0$ on $\overline{\Omega}$.

The classical Keller-Segel system of chemotaxis model was introduced by Keller and Segel [9] in 1970 to describe the cells (with density u) move towards the concentration gradient of a chemical substance v produced by the cells themselves. Various forms of sensitivity

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functions can be chosen to model different types of chemotaxis mechanisms. Among them $\phi(v) = \frac{\chi}{v}$ was selected in (1.1) largely due to the Weber-Fechner's law for cellular behaviors, where the subjective sensation is proportional to the logarithm of the stimulus intensity [6]. With $\phi(v) = \frac{\chi}{v}$, the cellular movements are governed by the taxis flux $\frac{\chi \nabla v}{v}$, which may be unbounded when $v \approx 0$. In the model (1.1), the values of the chemotactic sensitivity coefficient χ and the chemicals diffusion constant k play significant roles to determine the behavior of solutions. Obviously, either small $\chi > 0$ or large $k > 0$ benefits the global boundedness of solutions.

Recall the known results in the field with $k = 1$. At first consider the parabolic-elliptic analogue of (1.1), namely, the second parabolic equation in (1.1) is replaced by the elliptic equation $0 = \Delta v - v + u$. It was known that all radial classical solutions are global-in-time if either $n \geq 3$ with $\chi < \frac{2}{n-2}$, or $n = 2$ with $\chi > 0$ arbitrary [11]. When $0 < \phi(v) < \frac{\chi}{v^l}$ with $l \geq 1$, $\chi > 0$, there is a unique and globally bounded classical solution if $\chi < \frac{2}{n}$ ($l = 1$) or $\chi < \frac{2}{n} \cdot \frac{l}{(l-1)^{(l-1)}} \gamma^{(l-1)}$ ($l > 1$), with $\gamma > 0$ depending on Ω and u_0 [3]. Next consider the parabolic-parabolic case. All solutions of (1.1) are global in time when either $n = 1$ [13], or $n = 2$ and $\chi < \frac{5}{2}$ under the radial assumption, while $\chi < 1$ under the non-radial assumption [1, 12]. For $n \geq 2$, (1.1) possesses global classical solutions if $0 < \chi < \sqrt{\frac{2}{n}}$, and moreover, $\chi < \sqrt{\frac{n+2}{3n-4}}$ ensures the global existence of weak solutions [19]. Certain radial weak solutions have been constructed when $\chi < \sqrt{\frac{n}{n-2}}$ [15]. Following, the global boundedness of solutions has been obtained for $\chi < \sqrt{\frac{2}{n}}$ [2]. Refer to [4, 10, 21] for the results on more general chemotaxis models.

Recently, under somewhat complicated conditions, Wang [16] established classical global solutions to the problem, a similar model to (1.1),

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \frac{\chi u}{v+c} \nabla v), & x \in \Omega, \quad t \in (0, T), \\ v_t = k \Delta v - \alpha v + \beta u, & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \quad t \in (0, T), \\ (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), & x \in \Omega, \end{cases}$$

with $\chi, c, k, \alpha, \beta > 0$. In the present paper, motivated by Winkler [19] and Fujie [2], we will prove the global existence-boundedness of classical solutions to (1.1), with simplified conditions. That is the following theorem.

Theorem 1 *Let $n \geq 2$, $u_0 \in C^0(\overline{\Omega})$, $v_0 \in W^{1,q}(\Omega)$ ($q > n$) with $u_0 \geq 0$, $v_0 > 0$ on $\overline{\Omega}$. Then, for any $k > 0$, there exists a global classical solution to (1.1), provided $\chi \in (0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}})$. Moreover, the solution is globally bounded under $n \leq 8$.*

Remark 1 Theorem 1 shows in what way the size of $k > 0$ (the diffusion strength of the chemicals v) effects the behavior of solutions to (1.1). It is interesting to observe that when

$n = 2$ the global existence-boundedness of solutions is independent of the size of $k > 0$, since $-\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}} \equiv 1$ with $n = 2$. Quite differently, when $n \geq 3$ the contribution of $k > 0$ is significant that the range of χ for global existence-boundedness of solutions will be enlarged (shrunk) as $k > 0$ is decreasing (increasing). The arbitrariness of $k > 0$ yields the “maximal” range with $\chi \in (0, 1)$ or the “minimal” range with $\chi \in (0, \frac{2}{n})$. That is to say for any $\chi \in (0, 1)$ (close to 1), there is $k > 0$ (small) such that the classical solution of (1.1) is globally bounded. On the other hand, for any $k > 0$ (large), there is $\chi \in (0, \frac{2}{n})$ to ensure the global boundedness. Finally, it is pointed out that if $k = 1$, the required range of χ in Theorem 1 becomes $0 < \chi < \sqrt{\frac{2}{n}}$, which coincides with those in [2, 19].

Remark 2 Now compare our results for the parabolic-parabolic chemotaxis model (1.1) with those for the corresponding parabolic-elliptic model, which can be thought as a special case of (1.1) with the diffusion constant of the chemicals v sufficiently large [3]. Just as mentioned in Remark 1, letting $k > 0$ be arbitrarily large results in the “minimal” permitted range with $0 < \chi < \frac{2}{n}$. This does agree those obtained for the parabolic-elliptic model in [11].

2 Preliminaries

In this section we introduce the local existence of classical solutions to (1.1) with required estimates involving χ and k , as well as some technical lemmas for the global boundedness as preliminaries.

Lemma 2.1 *Let $n \geq 2$, $u_0 \in C^0(\overline{\Omega})$, $v_0 \in W^{1,q}(\Omega)$ ($q > n$) with $u_0 \geq 0$, $v_0 > 0$ on $\overline{\Omega}$. Then, for any $k, \chi > 0$, there exists $T_{\max} \in (0, \infty]$, such that (1.1) has a unique nonnegative solution $u \in C^0([0, T_{\max}); C^0(\Omega)) \times C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ and $v \in C^0([0, T_{\max}); C^0(\Omega)) \times C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \times L_{\text{loc}}^\infty([0, T]; W^{1,q}(\Omega))$, where either $T_{\max} = \infty$, or $T_{\max} < \infty$ with $\lim_{t \rightarrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} = \infty$.*

Proof. For $k > 0$, it is known from Lemma 2.2 in [2] that there is $\eta > 0$, such that $\inf_{x \in \Omega} v(x, t) \geq \eta > 0$ for all $t > 0$. Consequently, the local existence lemma can be obtained by the classical parabolic theory, refer to Theorem 3.1 in [17]. \square

The following lemma is crucial to establish the global existence-boundedness conclusions of the paper. Denote $r_\pm(p) = (p-1)[\frac{p\chi(1-k)+2k}{p(k-1)^2+4k} \pm \frac{2\sqrt{k^2-p\chi k(k-1)-p\chi^2 k}}{p(k-1)^2+4k}]$. Throughout the paper, for simplicity, denote $T = T_{\max}$.

Lemma 2.2 *Let (u, v) solve (1.1) with $k > 0$ and $\chi \in (0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}})$. If*

$$\|v(\cdot, t)\|_{L^{p-r}(\Omega)} \leq c, \quad t \in (0, T) \quad (2.1)$$

with $p < \frac{k}{[\chi^2 - \chi(1-k)]_+}$, $r \in (r_-(p), r_+(p))$, and $c > 0$, then

$$\int_{\Omega} u^p v^{-r} dx \leq \tilde{c}, \quad t \in (0, T) \quad (2.2)$$

with some $\tilde{c} > 0$.

Proof. It is known via a simple computation with (1.1) that

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} u^p v^{-r} dx &= p \int_{\Omega} u^{p-1} v^{-r} [\Delta u - \chi \nabla \left(\frac{u}{v} \nabla v \right)] dx - r \int_{\Omega} u^p v^{-r-1} (k \Delta v - v + u) dx \\
&= -p \int_{\Omega} \nabla(u^{p-1} v^{-r}) \cdot (\nabla u - \chi \frac{u}{v} \nabla v) dx + rk \int_{\Omega} \nabla(u^p v^{-r-1}) \cdot \nabla v dx \\
&\quad + r \int_{\Omega} u^p v^{-r} dx - r \int_{\Omega} u^{p+1} v^{-r-1} dx \\
&= -p(p-1) \int_{\Omega} u^{p-2} v^{-r} |\nabla u|^2 dx + [pr + prk + p(p-1)\chi] \int_{\Omega} u^{p-1} v^{-r-1} \nabla u \cdot \nabla v dx \\
&\quad - [r(r+1)k + pr\chi] \int_{\Omega} u^p v^{-r-2} |\nabla v|^2 dx + r \int_{\Omega} u^p v^{-r} dx - r \int_{\Omega} u^{p+1} v^{-r-1} dx \\
&\leq \int_{\Omega} \left[\frac{p[(p-1)\chi + r + rk]^2}{4(p-1)} - pr\chi - r(r+1)k \right] u^p v^{-r-2} |\nabla v|^2 dx \\
&\quad + r \int_{\Omega} u^p v^{-r} dx - r \int_{\Omega} u^{p+1} v^{-r-1} dx
\end{aligned}$$

by Young's inequality. Denote

$$f(r; p, \chi, k) = \frac{p[(p-1)\chi + r + rk]^2}{4(p-1)} - pr\chi - r(r+1)k,$$

and rewrite as the quadric expression in r

$$4(p-1)f(r; p, \chi, k) = [p(k-1)^2 + 4k]r^2 + [2p(p-1)\chi(k-1) - 4(p-1)k]r + p(p-1)^2\chi^2.$$

We know

$$\begin{aligned}
\Delta_r &= 4(p-1)^2[p\chi(k-1) - 2k]^2 - 4(p-1)^2p\chi^2[p(k-1)^2 + 4k] \\
&= 16(p-1)^2[k^2 - p\chi k(k-1) - p\chi^2 k] > 0
\end{aligned}$$

whenever $p < \frac{k}{[\chi^2 + \chi(k-1)]_+}$. Consequently, $f(r; p, \chi, k) < 0$ for any $r \in (r_-(p), r_+(p))$. This yields

$$\frac{d}{dt} \int_{\Omega} u^p v^{-r} dx \leq r \int_{\Omega} u^p v^{-r} dx - r \int_{\Omega} u^{p+1} v^{-r-1} dx, \quad t \in (0, T). \quad (2.3)$$

Due to $\int_{\Omega} u^p v^{-r} \leq \left(\int_{\Omega} u^{p+1} v^{-r-1} \right)^{\frac{p}{p+1}} \left(\int_{\Omega} v^{p-r} \right)^{\frac{1}{p+1}}$, we obtain (2.2) from (2.3) and (2.1). \square

Lemma 2.3 *Let (u, v) satisfy the second equation of (1.1) with $k > 0$, $1 \leq q \leq p \leq \infty$, $\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) < 1$. Then there exists $C > 0$, such that*

$$\|v(\cdot, t)\|_{L^p(\Omega)} \leq C(1 + \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^q(\Omega)}), \quad t \in (0, T) \quad (2.4)$$

Proof. Noticing $v(\cdot, t) = e^{t(k\Delta-1)}v_0 + \int_0^t e^{(t-s)(k\Delta-1)}u(\cdot, s)ds$ for $t > 0$, by the standard smoothing estimates for the heat semigroup under homogeneous Neumann boundary conditions [20], we can obtain for $q \leq p$ that

$$\begin{aligned} \|v(\cdot, t)\|_{L^p(\Omega)} &\leq \|e^{t(k\Delta-1)}v_0\|_{L^p(\Omega)} + \int_0^t \|e^{(t-s)(k\Delta-1)}u(\cdot, s)\|_{L^p(\Omega)}ds \\ &\leq C_1\|v_0(x)\|_{L^\infty(\Omega)} + C_2 \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^q(\Omega)} \int_0^t (1 + [k(t-s)])^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} e^{-(\lambda_1+\frac{1}{k})[k(t-s)]} ds \\ &\leq C_1\|v_0(x)\|_{L^\infty(\Omega)} + \frac{C_2}{k} \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^q(\Omega)} \int_0^\infty (1 + \alpha)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\lambda_1\alpha} d\alpha, \quad t \in (0, T), \end{aligned}$$

where $C_1, C_2 > 0$, λ_1 is the first nonzero eigenvalue of $-\Delta$ under the Neumann boundary condition. This proves (2.3). \square

Instead of the choice of $r = \frac{p-1}{2}$ in [2], we deal with the more complicated form $r = (p-1)\frac{[p\chi(1-k)+2k]}{p(1-k)^2+4k}$ to describe the effect of the chemicals diffusion rate k , and denote $h(p) \triangleq h(p; \chi, k) = \frac{p\chi(1-k)+2k}{p(1-k)^2+4k}$. For such $h(p)$, we have the following lemma.

Lemma 2.4 *Let $k > 0$, $\chi \in (0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}})$, and $p \in (1, \frac{k}{[\chi^2+\chi(k-1)]_+})$. Then $h(p) \in (0, 1)$.*

Proof. We have $h'(p) = \frac{2k(1-k)[2\chi-(1-k)]}{[p(1-k)^2+4k]^2}$.

If $k = 1$, then $h(p) \equiv \frac{1}{2}$.

If $k > 1$, then $h'(p) < 0$, and so

$$0 < \frac{\chi}{2\chi+k-1} = h(\frac{k}{\chi^2-\chi(1-k)} - 0) = h(\frac{k}{[\chi^2-\chi(1-k)]_+} - 0) < h(p) < h(1+0) = \frac{2k+\chi(1-k)}{(1-k)^2+4k} < \frac{2k}{(1+k)^2} < 1.$$

Now suppose $0 < k < 1$.

If $0 < \chi < \frac{1-k}{2}$, then $h'(p) < 0$, and so

$$0 < \frac{\chi}{1-k} = \lim_{p \rightarrow \infty} \frac{p\chi(1-k)+2k}{p(1-k)^2+4k} < h(p) < h(1+0) = \frac{2k+\chi(1-k)}{(1-k)^2+4k} < \frac{1}{2};$$

If $\chi = \frac{1-k}{2}$, then $h(p) \equiv \frac{1}{2}$;

If $\frac{1-k}{2} < \chi \leq 1-k$, then $h'(p) > 0$, and so

$$\frac{1}{2} < \frac{2k+\chi(1-k)}{(1+k)^2} = h(1+0) < h(p) < \lim_{p \rightarrow \infty} \frac{p\chi(1-k)+2k}{p(1-k)^2+4k} = \frac{\chi}{1-k} \leq 1;$$

If $1-k < \chi < \frac{-(k-1)+\sqrt{(k-1)^2+\frac{8k}{n}}}{2}$, then $h'(p) > 0$, and so

$$0 < \frac{1-k^2}{(1+k)^2} < \frac{2k+\chi(1-k)}{(1+k)^2} = h(1+0) < h(p) < h(\frac{k}{\chi^2-\chi(1-k)} - 0) = \frac{\chi}{2\chi+k-1} < 1.$$

The proof is complete. \square

Denote $c_0 = \inf_{p \in (1, \frac{n}{2}]} h(p)$ and $c^0 = \sup_{p \in (1, \frac{n}{2}]} h(p)$ with $n \geq 3$. By Lemma 2.4 and its proof, $c_0, c^0 \in (0, 1)$. The following lemma with c_0 and c^0 will play an important role for estimating the bound of u by the involved iteration in the next section.

Lemma 2.5 Let $k > 0$, $\chi \in (0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}})$, then $f(x) \triangleq \frac{n[(1-c^0)x + c^0 - c_0] + 2c_0x}{(n-2x)(1-c_0)} - x > 0$ for $x \in (1, \frac{n}{2}]$, provided $n \leq 8$.

Proof. It suffices to show $g(x) \triangleq 2(1-c_0)x^2 + [2c_0 - n(c^0 - c_0)]x + n(c^0 - c_0) > 0$ in $(1, \frac{n}{2}]$.

The case of $c_0 = c^0 \in (0, 1)$ is trivial.

Now suppose $0 < c_0 < c^0 < 1$. We have $\Delta_g = [2c_0 - n(c^0 - c_0)]^2 - 8n(1-c_0)(c^0 - c_0) = [n(c^0 - c_0)]^2 + (4c_0 - 8)n(c^0 - c_0) + 4c_0^2 < 0$, and hence $g(x) > 0$, whenever $4 - 2c_0 - 4\sqrt{1-c_0} < n(c^0 - c_0) < 4 - 2c_0 + 4\sqrt{1-c_0}$.

If $n(c^0 - c_0) \leq 4 - 2c_0 - 4\sqrt{1-c_0}$, then $\Delta_g > 0$. Due $n(c^0 - c_0) - 2c_0 < 4(1-c_0) - 4\sqrt{1-c_0} < 0$, we know that both the two roots of $g(x)$ must be negative. With $g(1+0) = 2$, we obtain $g(x) > 0$.

If $n(c^0 - c_0) \geq 4 - 2c_0 + 4\sqrt{1-c_0}$, then $\Delta_g > 0$, and the two roots of $g(x)$ satisfy $x_2 \geq x_1 > 0$. Together with $g(1+0) = 2$ and $g(\frac{n}{2}) = \frac{n^2}{2}(1-c^0) + nc_0 > 0$, the positivity of $g(x)$ for $x \in (1, \frac{n}{2}]$ requires that the minimal point of $g(x)$ satisfies $\frac{n(c^0 - c_0) - 2c_0}{4(1-c_0)} < 1$, i.e., $n(c^0 - c_0) + 2c_0 < 4$, by Vieta's formulas. This contradicts the case $n(c^0 - c_0) \geq 4 - 2c_0 + 4\sqrt{1-c_0}$. So, the case itself should be excluded to ensure $g(x) > 0$ in $(1, \frac{n}{2}]$. Rewrite the case as $c^0 \geq \frac{(n-2)c_0 + 4 + 4\sqrt{1-c_0}}{n} \triangleq \alpha(c_0)$ with $c_0 \in (0, 1)$. We have $c^0 \geq \alpha(c_0) > \min\{\alpha(0), \alpha(1)\} = \min\{\frac{n+2}{n}, \frac{8}{n}\}$. Thus, we would get a contradiction that $c^0 > 1$, whenever $n \leq 8$. \square

3 Proof of main result

We deal with the proof of the main result of the paper in this section.

Proof of Theorem 1.

We at first show that the local solutions ensured by Lemma 2.1 should be global.

Assume $p > q$. By the Hölder inequality, we have

$$\int_{\Omega} u^q dx = \int_{\Omega} (u^p v^{-r})^{\frac{q}{p}} v^{\frac{rq}{p}} dx \leq \left(\int_{\Omega} u^p v^{-r} dx \right)^{\frac{q}{p}} \left(\int_{\Omega} v^{\frac{rq}{p-q}} dx \right)^{\frac{p-q}{p}}, \quad 0 < t < T. \quad (3.1)$$

Let $p < \frac{k}{[\chi^2 + \chi(k-1)]_+}$. We know from (2.3) in the proof of Lemma 2.2 with $r \in (r_-(p), r_+(p))$ that

$$\frac{d}{dt} \int_{\Omega} u^p v^{-r} dx \leq r \int_{\Omega} u^p v^{-r} dx, \quad t \in (0, T),$$

and hence

$$\int_{\Omega} u^p v^{-r} dx \leq C, \quad t \in (0, T) \quad (3.2)$$

with $C = C(t) > 0$. Notice $\chi \in (0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}})$ with $p < \frac{k}{[\chi^2 + \chi(k-1)]_+}$ admits $p > \frac{n}{2}$.

Take $q \in (\frac{n}{2}, p) \subset (1, \min\{p, \frac{n(p-r)}{[n-2r]_+}\})$. Then $\frac{n}{2}(\frac{1}{q} - \frac{p-q}{rq}) < 1$. By Lemma 2.3, we have with $C_1 > 0$ that

$$\|v\|_{\frac{rq}{p-q}} \leq C_1(1 + \sup_{s \in (0,t)} \|u\|_q), \quad 0 < t < T. \quad (3.3)$$

Combining (3.1)–(3.3) yields

$$\int_{\Omega} u^q dx \leq C_2(1 + (\sup_{s \in (0,t)} \int_{\Omega} u^q dx)^{\frac{r}{p}}), \quad t \in (0, T),$$

and hence $\int_{\Omega} u^q dx \leq C_3$ for $t \in (0, T)$, where $C_2 = C_2(t) = \tilde{C}_2 e^{rt}$ with $\tilde{C}_2 > 0$, $C_3 = C_3(t) > 0$, and $\frac{r}{p} < \frac{p-1}{p} < 1$ by Lemma 2.4. Now we can follow the proof of Lemma 3.4 in [19] to obtain the global existence of solutions for (1.1).

Next, we prove the solutions established above are also globally bounded if $n \leq 8$. We should verify that $\int_{\Omega} u^q dx < C$ with some $q > \frac{n}{2}$ and $C > 0$ [19].

The case of $n = 2$ is simple. Take

$$\begin{cases} p \in (1, \frac{k}{[\chi^2 + \chi(k-1)]_+}), \\ r = (p-1)h(p). \end{cases}$$

Then $1 - \frac{1}{p-r} < 1$. By Lemma 2.3 with the L^1 -conservation of u , we have

$$\|v\|_{L^{p-r}} \leq C_4, \quad t \in (0, T)$$

with time-independent $C_4 > 0$. By Lemma 2.2,

$$\int_{\Omega} u^p v^{-r} dx \leq C_5, \quad t \in (0, T) \quad (3.4)$$

with some $C_5 > 0$. We deduce from (3.1), (3.3) and (3.4) that there exists $q > 1$, such that $\|u\|_q \leq C_6$ for $t \in (0, T)$ with $C_6 = C_6(k) > 0$.

Now consider the case of $3 \leq n \leq 8$. We should fix the assumption (2.1) with some $p > \frac{n}{2}$ and the constant independent of t there. We will do it via an iteration procedure based of Lemma 2.5.

1° Take

$$\begin{cases} p_0 \in (1, \min\{\frac{k}{[\chi^2 + \chi(k-1)]_+}, \frac{n(1-c_0) + 2c_0}{(n-2)(1-c_0)}\}), \\ r_0 = (p_0 - 1)h(p_0). \end{cases} \quad (3.5)$$

Then $p_0 - r_0 \leq p_0 - (p_0 - 1)c_0 = (1 - c_0)p_0 + c_0 < \frac{n}{n-2}$, i.e., $\frac{n}{2}(1 - \frac{1}{p_0 - r_0}) < 1$. By Lemma 2.3 with the L^1 -conservation of u , we have

$$\|v\|_{L^{p_0 - r_0}} \leq C_7, \quad t \in (0, T),$$

and hence

$$\int_{\Omega} u^{p_0} v^{-r_0} dx \leq C_8, \quad t \in (0, T) \quad (3.6)$$

by Lemma 2.2, with some $C_7, C_8 > 0$. We know from (3.1), (3.3) and (3.6) that $\|u\|_{q_0} \leq C_9$ for $t \in (0, T)$ with $C_9 > 0$. If $p_0 > \frac{n}{2}$, take $q_0 \in (\frac{n}{2}, p_0) \subset (1, \min\{p_0, \frac{n(p_0-r_0)}{[n-2r_0]_+}\})$.

2° Assume $p_0 \leq \frac{n}{2}$. Take

$$\begin{cases} p_1 \in \left(p_0, \min\left\{\frac{k}{[\chi^2 + \chi(k-1)]_+}, \frac{n[(1-c^0)p_0 + c^0 - c_0] + 2c_0p_0}{(n-2p_0)(1-c_0)}\right\}\right), \\ r_1 = (p_1 - 1)h(p_1), \end{cases}$$

where the well-definedness of the interval for p_1 is ensured by Lemma 2.5. A simple calculation shows

$$p_1 - r_1 \leq (1 - c_0)p_1 + c_0 < \frac{n[(1-c^0)p_0 + c^0]}{n-2p_0} \leq \frac{n(p_0 - r_0)}{n-2p_0} = \frac{n \frac{n(p_0-r_0)}{n-2r_0}}{n - \frac{2n(p_0-r_0)}{n-2r_0}},$$

i.e., $\frac{n}{2}(\frac{1}{\frac{n(p_0-r_0)}{n-2r_0}} - \frac{1}{p_1-r_1}) < 1$. By Lemma 2.3 with $p_0 \leq \frac{n}{2}$, there is $q_0 \in (1, \frac{n(p_0-r_0)}{n-2r_0})$ such that

$$\|v\|_{p_1-r_1} < C_{10}, \quad t \in (0, T),$$

and thus

$$\int_{\Omega} u^{p_1} v^{-r_1} dx \leq C_{11}, \quad t \in (0, T) \quad (3.7)$$

by Lemma 2.2, with some $C_{10}, C_{11} > 0$. It follows from (3.1), (3.3) and (3.7) that $\|u\|_{q_1} \leq C_{12}$ for $t \in (0, T)$ with $C_{12} > 0$. If $p_1 > \frac{n}{2}$, take $q_1 \in (\frac{n}{2}, p_1) \subset (1, \min\{p_1, \frac{n(p_1-r_1)}{[n-2r_1]_+}\})$.

3° Assume $p_{l-1} \leq \frac{n}{2}$, $l = 1, 2, 3, \dots$. Take

$$\begin{cases} p_l \in \left(p_{l-1}, \min\left\{\frac{k}{[\chi^2 + \chi(k-1)]_+}, \frac{n[(1-c^0)p_{l-1} + c^0 - c_0] + 2c_0p_{l-1}}{(n-2p_{l-1})(1-c_0)}\right\}\right), \\ r_l = (p_{l-1} - 1)h(p_{l-1}), \end{cases} \quad (3.8)$$

where the interval for p_l is well defined due to Lemma 2.5. Repeat the procedure in 2°, we deduce with $q_l \in (1, \min\{p_l, \frac{n(p_l-r_l)}{[n-2r_l]_+}\})$ that $\|u\|_{q_l} \leq C_{13}$, $0 < t < T$, with some $C_{13} > 0$. Noticing $\frac{n[(1-c^0)p_{l-1} + c^0 - c_0] + 2c_0p_{l-1}}{(n-2p_{l-1})(1-c_0)} \rightarrow \infty$ as $l \rightarrow \infty$ by Lemma 2.5, we can realize $p_l > \frac{n}{2}$ after finite steps. Let $q = q_l \in (\frac{n}{2}, p_l)$ to get $\int_{\Omega} u^q dx \leq C_{14}$ for all $t \in (0, T)$ with $C_{14} > 0$. It is mentioned that for any fixed $k > 0$, the constants C_i , $i = 5, \dots, 14$, are all independent of $t \in (0, T)$ here.

Based on the above estimate $\|u(\cdot, t)\|_{L^q(\Omega)}$ with $q > \frac{n}{2}$, uniform for $t \in (0, T)$, we conclude the global boundedness of u by coping the related arguments in [19] for that to the case of $k = 1$. \square

Remark 3 Notice $p_0 < \frac{k}{[\chi^2 + \chi(k-1)]_+}$ in (3.5) with $\chi \in (0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}})$ admits $p_0 > \frac{n}{2}$. Moreover, a simple computation shows $p_0 < \frac{n(1-c_0)+2c_0}{(n-2)(1-c_0)}$ in (3.5) ensures $\frac{n(1-c_0)+2c_0}{(n-2)(1-c_0)} > \frac{n}{2}$ whenever $n = 3, 4$. Therefore, the Steps 2° and 3° in the proof of Theorem 1 are unnecessary for $n = 3, 4$ there. In addition, it should be pointed out that $c^0 = c_0 = \frac{1}{2}$ if $k = 1$, and thus the requirement $n \leq 8$ itself can be removed away for the global boundedness of solutions with $k = 1$.

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